De Numero Radicum in Æquationibus Solidis ac Biquadraticis, ſive tertiæ ac quartæ poteſtatis, earumq; limitibus, tractatulus Authore *E. Halley*.

CUM in tractatulo, quem nuper publici juris feci in actis hiſce Philoſophicis, Num. 188; *Methodum aperuiſſem, qua Problemata ſolida utcunq; affecta minimo negotio, unica data Parabola & Circulo, ſimpliciſſime conſtrui poſſint; ſub finem mihi ſeſe obtulit contemplatio jucunda ſatis, nempe ex his Conſtructionibus Numerum radicum in quavis Æquatione, earumq; Limites ac ſigna facile conſequi ac determinari: quocirca fidem dedi me brevi de hac materia diſſertatiunculam aliquam ſcripturum, in qua ſi non Principibus, ſaltem ſecundæ claſſis Geometris, me non ingratum nec inutile præſtiturum omnino perſuaſum habui.*

Propius vero inſpicienti mihi compertum eſt, me imprudentem inter ardua Geometrica illapſum, ac jam iis tractandis deſignatum, quibus olim laboravere Viri illuſtres Harriottus *noſtras, ac* Carteſius; *in quibus pari fato utriq; Paralogiſmum, (forſan in eorum ſcriptis Geometricis unicum) diverſo tamen modo, admiſere; uti poſthac probabitur: ſed* Quandoq; bonus dormitat. *Qua propter agnita rei tum difficultate tum præſtantia, totis viribus incumbere ſtatui, ne promiſſis exequendis impar crederer, ac ne Geometriæ pars tam eximia, tamq; parum culta, diutius tenebris involuta lateret; ſed ope noſtra lucide his paucis expoſita daretur.*

Imprimis vero Lectorem monitum velim, quod dum his legendis operam dat, oportet prædictam diſſertationem Num. 188. *editam, ad manum habere, ac Conſtructiones ibidem traditas probe callere; quia quæ ſequuntur ab illis maxima ex parte pendent, quas tamen hic repetere vix integrum eſſet.*

Ex Carteſio *& ex ibi dictis conſtat, tam in Cubicis quam in Biquadraticis æquationibus, radices exponi poſſe demittendo perpendicula in Axem, datamve diametrum Parabolæ datæ, ab interſectionibus Curvæ illius cum Circulo. Cumq; Circulus Parabolam ſecans, vel in quatuor vel duobus punctis eam interſecare neceſſe eſt, conſtat in Biquadraticis vel duas vel quatuor radices veras, Affirmativas vel Negativas, ſemper haberi; uti etiam ſi forte Circulus illam tangat, quo in caſu æqualitas duarum radicum ejuſdem ſigni concluditur. In Cubicis autem, quoniam una ex interſectionibus ad Conſtructionem requiritur, non niſi una vel tres reliquæ radices deſignant unam vel tres; uti in Caſu contactus, unde conſtat duas æquales reperiri Radicis, Problemaq; unde reſultat æquatio revera planum eſſe.*

Cubicæ itaq; omnes quomodocunq; affectæ una vel triplici radice explicabiles ſunt, utiq; ſemper poſſibiles, nempe ſi radices Negativas pro veris admiſeris: ſic Biquadraticæ, quarum terminus ultimus r *ſigno — affecta eſt, duabus vel quatuor. Aſt ſi habeatur* + r *in æquatione, eaq; tanta ſit, ut* $\sqrt{\mathrm{GDq} - \mathrm{ar}}$, (*vide* fig. pag. 341.) *minor ſit quam ut Circulus, eo Radio ac centro* G *deſcriptus, Parabolam contingere in aliquo puncto poſſit, æquatio data omnino impoſſibilis eſt, nec ulla Radice Negativa vel Affirmativa explicabilis: Sed de his plura in ſequentibus.*

Quoniam vero tanta intercedit differentia inter caſus Cubicarum & Biquadraticarum, ut ſimul comprehendi nequeant; primum Cubicas deinde alteras tractabimus. Cubicæ vero infinitis Circulis in data Parabola conſtruuntur, Biquadraticæ autem unico tantum (*ſaltem his methodis*)*: id adeo quia ponendo* $z - e$ *ſive indeterminata aliqua, æqualem nihilo, æquatio Cubica reducitur ad Biquadraticam eaſdem radices cum Cubica habentem, atq; inſuper aliam ipſi* e *æqualem; unde fit ut tot Circulis diverſis conſtrui poſſit Cubica, quot imaginari velis quantitates* e, *id eſt infinitis. Inter has vero Conſtructiones illa, quam dedi* (pag. 342.) *longe facillima eſt. Huic tamen non multum cedit alia, quæ ad enucleationem Numeri Radicum, earumq; limi-*

tum magis accommodata videtur, quæq; ortum trahit ex ablatione ſecundi termini, ponendo modo vulgari x = z + *vel* — *tertia parte Coefficientis termini ſecundi. Hæc autem eſt. Data Parabola* ABY (*Fig.* I.) *ejuſq; Vertice* A, *axe* AE *&* *Latere recto* a, *reducatur æquatio ad formam conſuetam,* viz. $z^3.\ b\,z^2.\ a\,p\,z.\ a\,a\,q. = 0.$ *Deinde ad diſtantiam* $\frac{1}{3}$b *ducatur Axi parallela* BK, *dextrorſum quidem ſi fuerit* + b, *aliter ſiniſtrorſum, Parabolæ occurrens in* B; *ac lineæ ſuppoſitæ* AB *erigatur perpendicularis utrinque interminata* DP, *axi occurrens in puncto* G. *De* B *in Axem demitte perpendiculum* BC, *&* *ipſi* AC *fiat* GE *ſemper æqualis, ac verſus inferiora ponatur. Ab* E *fiat* EH = $\frac{1}{2}$ p, *ſurſum quidem, ſi in æquatione fuerit* + p, *deorſum vero ſi* — p, *ac e puncto* H (*vel ex* E *ſi defuerit quantitas* p) *educatur perpendiculum* HQ *interminatæ* DP *occurrens in puncto* O. *Denique in lineæ* HQ *interminata, fiat* OR = $\frac{1}{2}$ q, *ab* O *dextrorſum ſi fuerit* — q, *ſiniſtroſum ſi* + q, *collocanda: ac Circulus centro* R, *radio* RA *deſcriptus, tot punctis ſecabit Parabolam, quot æquatio propoſita veras habet radices; eæq; erunt perpendicula* ZY *a punctis interſectionum* Y *in axi parallelam* BK *demiſſa; quarum quæ ad dextram lineæ* BK *Affirmativæ ſunt, ad ſiniſtram Negativæ.*

Hujus Conſtructionis commoditas in eo conſiſtit, quod circulo per Verticem tranſeunte peragitur, perinde ac ſi defuiſſet ſecundus Terminus; ideoq; ad Radicum Numerum determinandum, ſufficit Loci ſive Lineæ Curvæ proprietates perſpectas habere, quæ ſpatia diſcriminat, ubi ſi ponatur centrum Circuli qui per Parabolæ Verticem tranſeat, circumferentia ejus vel uno vel tribus aliis punctis eam ſecabit; hoc eſt Lineæ curvæ, in quam incidunt centra omnium Circulorum per verticem tranſeuntium ac deinde Parabolam tangentium, naturam definire.

Locus autem ille eſt Parabolois, quam cum Cl. Walliſio *ſemicubicalem appellare licet, ſive in qua Cubi applicatorum ad Axem ſunt inter ſe ut Quadrata portionum Axis. Cujus Latus rectum eſt* $\frac{27}{8}$ *Lateris recti datæ Parabolæ, Vertex vero punctum* V (Fig. I.) *exiſtente* AV *dimidium lateris recti ejuſdem Parabolæ. Hoc eſt, ſi ponatur Unitas pro latere recto datæ Parabolæ,*

bolæ, $\frac{8}{27}$ *cubi ordinatim applicatæ æquabuntur quadrato partis diametri, ſive cubus ex* $\frac{2}{3}$ V H *quadrato ex* H R, *ſi ſcil.* R *ſit centrum circuli qui per verticem Parabolæ tranſeat eamq; deinde contingat; Hæc eſt Curva illa quam primus mortalium* Nelius *Noſtras rectæ datæ æqualem demonſtravit, eaq; occaſione apud Principes Geometras dudum celebris; ejuſq; proprietates* Cl. Walliſius *ſub finem Libri de Ciſſoide, &* Hugenius prop. 8 *&* 9 de linearum Curvarum evolutione, *aliiq; acri ingenio diſquiſivere, Quorum ſcripta conſulat Lector. Hæc Curva utrinq; ab Axe Parabolæ deſcripta,* viz. V N L, V P X, *ſpatium complectitur, in quo ſi ponatur centrum Circuli, qui per verticem* A *tranſeat, interſecabit ille Parabolam in tribus aliis punctis; Spatia vero ab Axe remotiora centra præbent circulis non niſi uno præter verticem puncto Parabolam ſecantibus.*

His probe intellectis jam ad determinandum Radicum numerum accingimur: Ac primum deficiat ſecundus terminus; ſitque Latus rectum 1, *vel* A V $= \frac{1}{2}$; *In conſtructione* V H *eſt* $\frac{1}{2}$ p, H R *vero* $\frac{1}{2}$ q; *cumq; ſi fuerit* $+$ p, *ab* V *verſus ſuperiora ponendo ſit* $\frac{1}{2}$ p, *centrum circuli extra ſpatium* L V X *ſemper conſtituitur; ideoq; una tantum radice explicabilis eſt, affirmativa ſi* $-$ q, *negativa ſi* $+$ q: *quæ quidem radices* Cardani *Regulis inveſtigantur. Si vero fuerit* $-$ p, V H $= \frac{1}{2}$ p *inferne ponitur, ac fieri poteſt ut* H R *cadat inter Axem & Curvam* V X *vel* V L, *ſi ſcilicet Cubus ex* $\frac{2}{3}$ V H, *ſive ex* $\frac{1}{3}$ p, *major ſit quam quadratum ex* $\frac{1}{2}$ q, *ſive* $\frac{1}{27}$ p p p *major quam* $\frac{1}{4}$ q q, *quo in caſu tres dantur radices, duæ Negativæ, ſi fuerit* $-$ q, *ac una Affirmativa earum ſummæ æqualis; vel ſi* $+$ q, *duæ Affirmativæ unaq; Negativa. Quod ſi* $\frac{1}{27}$ p p p *minor ſit quam* $\frac{1}{4}$ q q, *una tantum reperitur Radix, Affirmativa ſi* $-$ q, *negativa ſi* $+$ q. *Atq; hæc paſſim docentur ab iis qui hanc Geometriæ partem tractarunt.*

Jam adſint omnes termini, ac primum proponatur, Exempli cauſa, æquatio hæc $z^3 - z^2 b + z p - q = 0$; *cui etiam Figuram* I. *adaptavimus. In hujus conſtructione* B C $= \frac{1}{3}$b, V G $= \frac{1}{2}$ A C $= \frac{1}{18}$bb, V E bb, V H $\frac{1}{6}$bb $- \frac{1}{2}$p, G H $\frac{1}{9}$bb $- \frac{1}{2}$p *vel* $\frac{1}{2}$p $- \frac{1}{9}$bb, *hinc* H O $= \frac{1}{27}$b^3 $- \frac{1}{6}$b p *vel* $\frac{1}{6}$b p $- \frac{1}{27}$b^3, *atq;* H R *ſive diſtan-*

tia Centri circuli R *ab Axe, eſt ſemper differentia inter* $\frac{1}{6}bp$ *&* $\frac{1}{27}b^3 + \frac{1}{2}q$; *quæ ſi æquantur, centrum cadit in Axe; ſi* $\frac{1}{6}bp$ *major ſit quam* $\frac{1}{27}b^3 + \frac{1}{2}q$ *ad ſiniſtram Axis, ſin minor ad dextram. Si itaq; Cubi ex* $\frac{1}{3}$VH, (*hoc eſt ex* $\frac{1}{9}bb - \frac{1}{3}p$ *quam nominemus* d) *Latus quadratum ſive* $\sqrt{ddd}$, *majus ſit quam* HR, *ſive differentia inter* $\frac{1}{27}b^3 + \frac{1}{2}q$ & $\frac{1}{6}bp$; *reperitur centrum* R *intra ſpatium* N P V, *Paraboloidibus* V P X, V N L *ac recta interminata* D N P *circumſcriptum: ac proinde circulus Parabolam ſecabit in tribus punctis* Y, Y, Y, *ad dextram lineæ* B K *ſitis, atq; adeo æquatio tres habet radices Affirmativas. Centro vero extra hoc ſpatium* N V P *conſtituto, non niſi una radice Affirmativa explicari poteſt. Hic obiter notandum rectam* D P *Paraboloidem* V P X *tangere in puncto* P, *exiſtente* E P $\frac{1}{27}b^3$; *alteram vero* V N L *ſecare in puncto* N, *ita ut demiſſo in axem Perpendiculo* N F, V F *ſit pars quarta ipſius* E V *ſive* $\frac{1}{24}bb$, N F *vero* $\frac{1}{108}b^3$. V W *autem, quæ e puncto* V *axi perpendiculariter erecta lineæ* D P *occurrit in* W, *æqualis eſt* $\frac{1}{54}bbb$ *ſive* $\frac{1}{2}$E P.

Hinc tuto concluditur ſi in æquatione vel p *major ſit quam* $\frac{1}{3}bb$, *vel* q *major quam* $\frac{4}{27}b^3$, *non niſi unam eamq; affirmativam radicem reperiri; Fallit itaq; Regula* Carteſii (*Edit.* Amſt. 1659 *pag.* 70) *ubi tot veras dari radices quot ſunt in æquatione mutationes ſignorum* + & — *pronunciat, fruſtra etiam in Commentariis ſuis Sphalma hoc excuſante* Schootenio; *Fingi enim poſſunt infinite plures æquationes præcedentis formulæ tres ſignorum mutationes habentis, quæ unam tantum quam quæ tres habeant radices. Propoſitio etiam quinta Sectionis quintæ Artis Analyticæ* Harriotti *Noſtri, uti* Prob. 18 *Numeroſæ Poteſt. Reſol.* Vietæ, *vix ſatis firma eſt, cum ex limitationibus quas ibi poſuerunt, toti parallelogrammo* P I V W *id conveniat, quod ſoli ſpatio* N V P *jam competere probavimus, hoc eſt ut centrum præbeat circulo tribus aliis punctis præter verticem Parabolam ſecante.*

Quantitas autem q, *ſive terminus ult., datis* b & p, *ea lege ut* p *minor ſit quam* $\frac{1}{3}bb$, *accurate limitatur ex præcedente æquatione* $\sqrt{ddd} = \frac{1}{27}b^3 + \frac{1}{2}q \backsim \frac{1}{6}bp$; *cum ſcil. Circulus Parabo-*

lam contingat. Itaq; $\frac{1}{2}q$ minor eſſe debet quam $\frac{1}{6}bp - \frac{1}{27}b^3 + \sqrt{ddd}$; at ſi p major fuerit quam $\frac{1}{4}bb$, majorem etiam eſſe oportet $\frac{1}{2}q$ quam $\frac{1}{6}bp - \frac{1}{27}b^3 - \sqrt{d^3}$ ne cadat centrum in ſpatiolo N V W: Atq; his conditionibus æquatio ſemper triplici radice explicabilis erit, aliter non niſi una. Semper vero, ſive tres ſive una, Affirmativæ ſunt, ob poſitionem centri R ad dextram lineæ D P.

Atq; hic eſt caſus maxime difficilis, ita ut quicunq; præmiſſa bene calleat ſequentia facili negotio intelliget. Detur jam æquatio $z^3 - bz^2 + pz + q = 0$. Hic ut tres habeantur radices, oportet centrum circuli alicubi intra ſpatium P N Δ, rectis P N, P Δ & curva Paraboloidis N Δ definitum, reperiri; quapropter cum E F ſit $= \frac{1}{3}bb$, p minor eſſe debet quam $\frac{1}{4}bb$; jam ad determinationem quantitatis q, exiſtente $d = \frac{1}{9}bb - \frac{1}{3}p$ ut antea, $\sqrt{ddd} + \frac{1}{27}bbb - \frac{1}{6}bp$ ſemper major eſſe debet quam $\frac{1}{2}q$, ut conſtituatur centrum circuli in ſpatio prædicto P N Δ: quod cum ſit æquatio talis duas habet radices Affirmativas ac unam negativam. Si vero p major eſt quam $\frac{1}{4}bb$, uel $\frac{1}{2}q$ major quam $\sqrt{ddd} + \frac{1}{27}bbb - \frac{1}{6}bp$, non niſi una eaq; negativa radice explicabilis eſt.

Proponatur jam æquatio $z^3 - bz^2 - pz - q = 0$. Ut hæc æquatio tres habeat Radices, oportet centrum circuli alicubi inveniri in ſpatio indefinito, inter rectam D P D & curvam Paraboloidis P X; hic quantitas p non eſt obnoxia limitationibus, $\frac{1}{2}q$ vero ſemper minor eſſe debet quam $\sqrt{ddd} - \frac{1}{27}bbb - \frac{1}{6}bp$, poſito $d = \frac{1}{9}bb + \frac{1}{3}p$: Hoc pacto duæ dantur Radices Negativæ, ac una Affirmativa; aliter vero ſi $\frac{1}{2}q$ major ſit quam $\sqrt{ddd} - \frac{1}{27}bbb - \frac{1}{6}bp$, unica tantum Affirmativa exponi poteſt. Quarto loco ſit æquatio $z^3 - bz^2 - pz + q = 0$, quæ duas Affirmativas habet Radices ac unam Negativam ſi centrum circuli reperiatur in ſpatio indefinito inter rectas P Δ, P D ac curvam Paraboloidis Δ L; hoc eſt, (poſito $d = \frac{1}{9}bb + \frac{1}{3}p$,) ſi $\frac{1}{2}q$ minor ſit quam $\sqrt{ddd} + \frac{1}{27}bbb + \frac{1}{6}bp$; ſi vero $\frac{1}{2}q$ major hac quantitate fuerit, una tantum Negativa ineſt radix.

Quatuor autem æquationes reliquæ, in quibus habetur $+b$, quoad limitationem Numeri Radicum non differunt a prædictis, ſi ſig-

num termini ultimi mutetur, ſervato ſigno termini tertii; quæ vero Affirmativæ erant radices in illis hic fiunt Negativæ; & vice verſa. Sic in æquatione $z^3 - bz^2 + pz^2 - q = 0$. *Una vel tres erant Affirmativæ Radices; in hac vero* $z^3 + bz^2 + pz + q = 0$ *vel una uel tres Negativæ ſunt, ſub iiſdem conditionibus; nulla vero omnino Affirmativa. Sic in* $z^3 + bz^2 + pz - q = 0$, *duæ ſunt Negativæ & una Affirmativa, ſi* p *minor ſit quam* $\frac{1}{3}bb$, *ac* $\frac{1}{2}q$ *minor quam* $\sqrt{d^3} + \frac{1}{27}b^3 - \frac{1}{6}bp$, *quemadmodum in* $z^3 - bz^2 + pz + q = 0$ *duæ erant Affirmativæ & una Negativa; excedentibus autem leges præſcriptas* p *vel* q, *una tantum hic eſt radix Affirmativa, quæ ibi Negativa erat. Pari modo in* $z^3 + bz^2 - pz + q = 0$ *vel duæ ſunt Affirm. ac una Neg. vel una Negativa tantum. Deniq; iiſdem de cauſis in æquatione* $z^3 + bz^2 - pz - q$ *duæ ſunt Negativæ & una Affirm. vel una Affirm. tantum, quibus in æquatione* $z^3 - bz^2 - pz + q$ *duæ erant Affirm. & una Negativa, vel una Negativa tantum, nempe prout* $\frac{1}{2}q$ *major vel minor fuerit quam* $\sqrt{d^3} + \frac{1}{27}b^3 + \frac{1}{6}bp$.

Si deſuerit terminus tertius, ſive pz, *centrum* R *ſemper cadit in linea* IPEΔ, *quocirca ſi fuerit* $z^3 - bz^2 . * . - q$ *vel* $z^3 + bz^2 . * . + q$, *una tantum eſſe poteſt radix, ſi* $-b$ *Affirmativa, ſi* $+b$ *Negativa. At ſi fuerit* $z^3 - bz^2 . * . + q$ *vel* $z^3 + bz^2 . * . - q$, *duæ poſſunt eſſe Affirmativæ ac una Negativa in priore, vel una Affirm. & duæ Neg. in poſteriori, cadente centro in linea* PΔ *inter* P *ac* Δ, *hoc eſt ſi* $\frac{1}{4}q$ *minor ſit quam* $\frac{1}{27}b^3$; *ſin major fuerit, una tantum Negativa in priore, vel una Affirm. in poſteriore dari poteſt.*

Hactenus numerum radicum in Cubicis æquationibus plenius aſſecuti ſumus, reſtat ut nonnulla adjiciam de quantitate radicum. Hic primum notandum quod omnis æquatio tres habens radices ope Tabulæ Sinuum, Triſectione ſcilicet anguli, ſatis expedite reſolvi poſſit; ponendo ſcil. $\sqrt{\frac{4}{9}bb - \frac{4}{3}p}$ *vel* $\sqrt{4d}$, *ſi fuerit* $+p$ *in æquatione, vel* $\sqrt{\frac{4}{9}bb + \frac{4}{3}p}$, *ſi* $-p$, *pro Radio Circuli; Angulum vero triſecandum qui Sinum habeat in Tabula Sinuum* $\dfrac{\frac{1}{27}b^3 + \frac{1}{6}bp + \frac{1}{2}q:}{\sqrt{ddd}}$ *Invento hoc angulo, Sinus* *tertiæ partis ejus, ut & Sinus tertiæ partis compl. ad Semicirculum, eorumq; ſumma, ex Tabula Sinuum dabuntur. Hi vero Sinus in Radium* $\sqrt{\frac{1}{3}bb + p}$ *ducendi ſunt, & habebuntur quantitates* (y &, y &, y &, *in Fig.*) *quarum &* $\frac{1}{3}b$ *vel ſumma vel differentia, prout caſus poſtulat, veras radices Æquationis exhibebunt. Hæc omnia ex inventis* Cartesii *derivantur: Ut vero caſus omnes quantum fieri poſſit breviter complectar, dico quod centro* R, *in prima æquationum formula, cadente in ſpatio* V G P, *ſectiones duæ* Y, Y, *cadunt inter* A & B, *ac proinde utraq; ex minoribus radicibus minor eſt quam* $\frac{1}{3}b$, *tertia autem & major ſemper ſuperat* $\frac{1}{3}b$, *ſuperatur vero a* b. *Quod ſi cadat in ſpatio* G N V, *duæ majores ſunt quam* $\frac{1}{3}b$, *minores vero quam* $\frac{2}{3}b$, *tertia vero eſt* b — *duabus alteris, ac proinde minor quam* $\frac{1}{3}b$: *ſed adhibita limitatione quantitatis* p, *arctioribus terminis radicis includuntur. Maxima enim radix minor eſt quam* $\sqrt{\frac{4}{9}bb - \frac{4}{3}p} + \frac{1}{3}b$, *major vero quam* $\sqrt{\frac{1}{9}bb - p} + \frac{1}{3}b$; *at cum* $\frac{1}{4}b.b$ *minor eſt quam* p, *limes ille fit* $\sqrt{\frac{1}{9}bb - \frac{1}{3}p} + \frac{1}{3}b$. *Radix media ſemper minor eſt quam* $\sqrt{\frac{1}{4}bb - p} + \frac{1}{2}b$; *major vero quam* $\frac{1}{3}b - \sqrt{\frac{1}{9}bb - \frac{1}{3}p}$; *hunc vero limitem nunquam excedit radix minima, ſed cum quantitate* q *evaneſcit.*

In ſecunda formula præſcriptis legibus duæ ſunt affirmativæ ac una negativa, ac cadente centro in ſpatio G P E *altera ex affirmativis major eſt, altera minor quam* $\frac{1}{3}b$, *major vero non excedit* b, *Negativa autem major non eſſe poteſt quam* $\sqrt{\frac{1}{3}bb} - \frac{1}{3}b$, *eſt autem differentia ipſius* b *& ſummæ Affirmativarum. Centro autem in ſpatio* E N G △ *poſito, utraq; Affirmativa major eſt quam* $\frac{1}{3}b$, *minor vero quam* $\sqrt{\frac{1}{2}bb} + \frac{1}{3}b$, *Negativa vero ſemper minor eſt quam* $\frac{1}{3}b$. *Limites autem propriores ex data* p *evadunt, radicis quidem maximæ Affirmativæ* $\sqrt{\frac{1}{4}bb - p} + \frac{1}{2}b$, *qua ſemper minor eſt, ut & major quam* $\sqrt{\frac{1}{9}bb - \frac{1}{3}p} + \frac{1}{3}b$; *hac tamen limite minor eſt altera Affirmativa, quæ cum quantitate* q *minuitur. Negativa vero ſemper minor eſt quam* $\sqrt{\frac{1}{3}bb - \frac{1}{3}p} - \frac{1}{3}b$, *ac deficiente quantitate* q *evaneſcit.*

In tertia formula duæ Negat. ſunt ac una Affirmativa: in hac, *ut & in quarta, Radices non limitantur a quantitate* b. *Affirmativa vero ſemper minor eſt quam* $\sqrt{\frac{4}{9}bb+\frac{4}{3}p}+\frac{1}{3}b$, *major tamen quam* $\sqrt{p+\frac{1}{4}bb}+\frac{1}{2}b$: *maxima vero ex Negativis ſemper major eſt quam* $\sqrt{\frac{1}{9}bb+\frac{1}{3}p}-\frac{1}{3}b$, *minor vero quam* $\sqrt{p+\frac{1}{4}bb}-\frac{1}{2}b$. *Minor autem ex Negativis ſemper minuitur cum minuta quantitate* q.

In quarta formula, cadente centro intra ſpatium L Δ P D; *ſi duæ ſint Affirmativæ ac una Negativa, maxima ex Affirmativis major eſſe nequit quam* $\sqrt{p+\frac{1}{4}bb}+\frac{1}{2}b$, *nec minor quam* $\sqrt{\frac{1}{9}bb+\frac{1}{3}p}+\frac{1}{3}b$; *minor vero radix ab hoc limite minuitur, minuta quantitate* q. *Negativa autem minor eſt quam* $\sqrt{\frac{4}{9}bb+\frac{4}{3}p}-\frac{1}{3}b$; *major vero quam* $\sqrt{p+\frac{1}{4}bb}-\frac{1}{2}b$.

Notandum vero hic radices Negativas ubiq; ſigno Affirmativo notari, quia hæ ſunt radices Affirmativæ quatuor æquationum illarum, in quibus habitur + b, *ac* q *ſigno contrario notatur; ut ſupra monui. Horum omnium demonſtratio ex eo conſequitur, quod ubicunq; centrum circuli* R *incidit in Lineas Curvas* V P X *vel* V Δ L, *circumferentia ejus Parabolam tangit in puncto, cujus diſtantia ab axe eſt* $\sqrt{\frac{1}{3}VH}$, *eamq; ſecat ex altera Axis parte, ad diſtantiam* $2\sqrt{\frac{1}{3}VH}$; *cum vero centrum cadit in lineam* D P D, *altera ex radicibus fit* $=0$, *ac proinde Cubica reducitur ad Quadraticam, ſive ad* $z^2-bz+p=0$ *cujus radices limites deſignant ubi evaneſcit quantitas* q: *ac quo minor eſt* q, *eo propius ad has limites accedunt radices. Quadratica eſt etiam cum centrum cadit in Axe; hoc eſt, cum* $\frac{1}{2}q=\frac{1}{6}bp-\frac{1}{27}b^3$ *in prima formula; vel* $\frac{1}{2}q=\frac{1}{27}bbb-\frac{1}{6}bp$ *in ſecunda; in tertia impoſſibile eſt; at in quarta cum* $\frac{1}{2}q=\frac{1}{27}bbb+\frac{1}{6}bp$; *quo in caſu minor ex Radicibus Affirmativis eſt* $\frac{1}{3}b$, *major* $\sqrt{\frac{1}{3}bb+p}+\frac{1}{3}b$; *Negativa vero* $\sqrt{\frac{1}{3}bb+p}-\frac{1}{3}b$. *In prima, Radices ſunt* $\frac{1}{3}b$ & $\frac{1}{3}b\pm\sqrt{\frac{1}{3}bb-p}$. *In ſecunda vero formula,* $\frac{1}{3}b$ & $\sqrt{\frac{1}{3}bb-p}+\frac{1}{3}b$ *ſunt Affirmativæ: Negativa autem* $\sqrt{\frac{1}{3}bb-p}-\frac{1}{3}b$.

Atq; hæc in Cubicis ſufficere poſſe videntur; ob eximium vero Uſum Methodi, qua ope Tabulæ Sinuum radices harum æquationum

onum inveniuntur, placuit unum vel alterum exemplum adjungere, ut praxis illius compendium inde innoteſcat. Proponatur Æquatio $zzz - 39zz + 479z - 1881 = 0$; *quæruntur radices* z. $\sqrt{\frac{1}{9}bb - \frac{1}{3}p} = \sqrt{9\frac{1}{3}} = \sqrt{d}$, *cujus duplum* $\sqrt{37\frac{1}{3}}$ *radius eſt Circuli; &* $\frac{\frac{1}{27}bbb + \frac{1}{2}q - \frac{1}{6}bp}{\sqrt{ddd}} = \frac{2197 + 940\frac{1}{2} - 3113\frac{1}{2}}{9\frac{1}{3}\sqrt{9\frac{1}{3}}}$, *ſive* $\frac{24}{9\frac{1}{3}\sqrt{9\frac{1}{3}}}$ *eſt ſinus Tabularis Anguli, hoc, eſt, facta diviſione ope Logarithmorum, Log.* 9.9251560, *cui reſpondet, Angulus* 57gr. 19m. 11s. ½. *Hujus tertia pars* 19g. 6m. 24s. *& complementi* 40g. 53m. 36s. *Sinus dant Log.* 9.514983. *&* 9.816011, *qui ducti in Rad.* $\sqrt{37\frac{1}{3}}$ *producunt* Y &, *&* Y & *Log.* 0.301030 = 2 *& Log.* 0.601059 = 4, *tertia vero* Y & *æqualis eſt eorum ſummæ ſive* 6. *Ideoq; radices ſunt* $13 - 4 = 9$. $13 - 2 = 11$ *&* $13 + 6 = 19$, *ex quibus ſingulis conflatur prædicta æquatio. Ubi Notandum duas minores radices non excedere* $\frac{1}{3}b$ *vel* 13, *quia centrum* R *in conſtructione cadit ad dextram Axis; id eſt* $\frac{1}{6}bp$ *minor eſt quam* $\frac{1}{27}b^3 + \frac{1}{2}q$.

Exemplum alterum ſit $x^3 - 15x^2 - 229x - 525 = 0$. *& quærantur radices.* $\sqrt{\frac{1}{9}bb + \frac{1}{3}p} = \sqrt{101\frac{1}{3}} = \sqrt{d}$, *& Radius Circuli* $\sqrt{405\frac{1}{3}}$. $\frac{\frac{1}{27}b^3 + \frac{1}{6}bp + \frac{1}{2}q}{\sqrt{ddd}} = \frac{125 + 572\frac{1}{2} + 262\frac{1}{2}}{101\frac{1}{3}\sqrt{101\frac{1}{3}}} = \frac{960}{101\frac{1}{3}\sqrt{101\frac{1}{3}}}$ = *Sinui Tabulari Arcus, cujus Log.* 9.9736426, *& Arcus ipſe* 70gr. 14m. 22s. *hujus pars tertia eſt* 23gr. 24m. 47½s. *& complementi* 36. 35. 12½; *quorum Sinus Log. ſunt* 9.599183 *&* 9.775275, *quibus addito Log.* $\sqrt{405\frac{1}{3}}$ *fiunt Log.* 0.903089 = 8 *& Log.* 1.079181 = 12, *& eorum ſumma* = 20. *Hinc concluditur* $20 + \frac{1}{3}b$ *vel* 25 *æquari radici Affirmativæ, &* 8 *&* $12 - \frac{1}{3}b$ *ſive* 3 *&* 7 *Negativis. Quod ſi æquatio fuiſſet* $x^3 + 15x^2 - 229x + 525 = 0$, 3 *&* 7 *fuiſſent Affirmativæ;* 25 *vero Negativa. Cæteræ autem Cubicæ unica tantum Radice explicabiles juxta Regulas* Cardani *reſolvendæ ſunt, poſtquam demptus fuerit ſecundus terminus; nec video quo pacto minori calculo hoc negotium peragi poſſit. At ſi deſideretur radix hæc in Quantitatibus* b, p, q *expreſſa, dico* *eam eſſe in prima formula,* $\frac{1}{3}b +$ *vel* $-$ *ſumma vel differentia Radicum Cubicarum ex* $\sqrt{\frac{1}{4}qq - \frac{1}{108}p^2b^2 + \frac{1}{27}b^3q - \frac{1}{6}bpq + \frac{1}{27}p^3} \pm \frac{1}{27}b^3 + \frac{1}{2}q - \frac{1}{6}bp$: *viz.* $+$, *ſi* $\frac{1}{27}b^3 + \frac{1}{2}q$ *major ſit quam* $\frac{1}{6}bp$, *aliter* $-$; *Summa vero quoties* $\frac{1}{3}bb$ *major eſt quam* p; *ſin minor fuerit* $\frac{1}{3}bb$, *differentia. Inq; cæteris formulis radix ſemper conflatur ex iiſdem elementis, variatis tamen ſignis* $+$ *&* $-$, *ut facile percipiet qui velit experiri.*

Ope vero Tabulæ Logarithmicæ Sinuum Verſorum Radices hæ ſatis prompte inveniuntur; nempe ſi Coefficientes Numeri ſint Surdi vel fracti, ac radices Numeris ineffabiles; ut plerumq; fit. Hæc autem eſt Regula: In prima ac ſecunda formula, ſi $\frac{1}{3}bb$ *minor ſit quam* p; *ſit* $\frac{1}{3}p - \frac{1}{9}bb = d$, *& poſita differentia inter* $\frac{1}{6}bp$ *&* $\frac{1}{27}b^3 + \frac{1}{2}q$, *hoc eſt* HR, *in prima, ac inter* $\frac{1}{6}bp + \frac{1}{2}q$ *&* $\frac{1}{27}b^3$, *in ſecunda, pro Radio; inveniatur angulus cujus Tangens eſt* $d\sqrt{d}$. *Deinde ut Co-ſinus hujus anguli, ad ejuſdem Sinum verſum: ita differentia pro Radio habita, ad quartum; cujus Latus cubicum triſecando Logarithmum habebitur: ac diviſo* $\frac{1}{3}p - \frac{1}{9}bb$ *per hoc Latus Cub. e Quoto ſubducatur Diviſor: Reſiduum erit quantitas* Y&, *in Fig.* I. *Hujus Reſidui ac* $\frac{1}{3}b$ *ſumma, ſi centrum cadit ad dextram Axis, aliter differentia earundem, Radix erit quæſita. Quod ſi* $\frac{1}{3}bb$ *major ſit quam* p, *poſito* HR *pro Radio, ſit* $d\sqrt{d}$ *ſive diſtantia Paraboloidis ab Axe, Sinus Arcus cujuſdam; Hujus Sinus verſus ducatur in Radium, ſive* $\frac{1}{6}bp - \frac{1}{27}b^3 \pm \frac{1}{2}q$, *ac triſecto producti Logarithmo, habebitur ejus Latus Cubicum, per quod dividatur* $\frac{1}{9}bb - \frac{1}{3}p$. *dico Quoti ac diviſoris ſummam eadem Lege additam vel ablatam ex* $\frac{1}{3}b$, *Radicem quæſitam exhibere. Ac par eſt ratio in tertia ac quarta formulis, niſi quod* $\frac{1}{27}bbb + \frac{1}{6}bp \pm \frac{1}{2}q$ *pro Radio aſſumenda eſt, ac* $\frac{1}{9}bb + \frac{1}{3}p$ *in* $\sqrt{\frac{1}{9}bb + \frac{1}{3}p}$ *ſive* $d\sqrt{d}$ *pro Sinu: Sed hæc præcepta exemplis fortaſſe melius percipientur.*

Sit æquatio Cubica $zzz - 17zz + 54z - 350$, *ac quæratur Radix* z: *Hic* $\frac{1}{3}bb$ *major eſt quam* p, *ſed* q *major eſt quam Cubus ex* $\frac{1}{3}b$, *ideoq; una tantum Affirmativa Radice explicabilis eſt. Jam* $\frac{289}{9} - \frac{54}{3}$ *eſt* d, *ac* $\frac{127}{9}\sqrt{\frac{127}{9}}$ *pro Sinu habenda eſt, ad Radium* $\frac{4913}{27} + 175 - 153$, *hoc eſt* $\frac{5507}{27}$:

Arcus vero competens ſit 15gr. 3m. 49s. *Hujus Sinus Verſi Log.* 8.5362376. *additus Log. Radii* 2.3095913. *dat.* 0.8457889. *cujus tertia pars* 0.2819276. *eſt Log. Radicis Cubicæ* 1.91394, *quo diviſore diviſo* $\frac{127}{9}$ *ſive* d, *ſit Quotus* 7.37281; *Quoti ac diviſoris ſumma, aucta additione* $\frac{1}{3}$b, *ſit Radix quæſita, nempe* 14.9534, &c.

Exactis Cubicis Biquadraticas jam aggrediamur; Hæ ſemper vel nullam, vel duas, vel quatuor Radices veras habent, quarum determinatio partim a Coefficientibus, partim a ſigno & magnitudine numeri abſoluti dati, pendet; Harum omnium Conſtructionem generalem (in N°. 188. Pag. 341) *ſatis concinnam prodidi, quam Lector jam vidiſſe ſupponitur; Figuram tamen eo ſpectantem* (Fig. II.) *huc transferre viſum eſt. In Conſtructione æquationis* $z^4 - bz^3 + pzz - qz + r = 0$, *ſit* $BD = \frac{1}{4}b$, $AB = \frac{1}{16}bb$, $BK = \frac{1}{2}$, *ſive dimidio Lateris recti,* $KC = 2\,AB = \frac{1}{8}bb$. $KE = \frac{1}{8}bb - \frac{1}{2}p$, $AE - \frac{1}{2} = \frac{3}{16}bb - \frac{1}{2}p$ $FE = \frac{1}{16}b^3 - \frac{1}{4}bp$, *ac* $EG = \frac{1}{16}b^3 - \frac{1}{4}bp + \frac{1}{2}q$; *quo facto Circulus centro* G, *Radio* $\sqrt{GD^2 - r}$, *interſecabit Parabolam vel nullo, duobus aut quatuor punctis, quæ perpendiculis in lineam* DH, *Radices omnes* z *exhibent. Ut autem quatuor ſint, evidens eſt centrum circuli alicubi conſtitui debere intra ſpatium, de cujus puncto quovis tria perpendicula in Curvam Parabolæ demitti poſſint; atq; ſimul radium minorem eſſe maximo ex illis perpendicularibus, majorem vero medio. Quod ſi centrum conſtituatur extra hoc ſpatium, ut non niſi una perpendicularis in Parabolam demitti poſſit, qua major ſit radius; vel ſi minor ſit media ex tribus perpend. major vero quam minima ex illis, duæ tantum poſſunt eſſe radices; nulla vero omnino datur, quoties radius* $\sqrt{GD^2 - r}$ *minor eſt minima ex tribus, vel una illa, quoties una tantum eſt.*

Jam quale ſpatium hoc ſit, quibuſq; limitibus diſcernitur, ac quibus conditionibus radius Circuli minor vel major ſit prædictis perpendicularibus, nobis reſtat inquirendum; ac primum quo pacto perpendicularis in Parabolam demitti poſſit oſtendendum eſt.

Sit ABC *Parabola*, AE *Axis ejus*, AV (Fig. III.) *ſemi-Latus rectum*, G *punctum de quo demittenda eſt perpendicularis: Ducatur Axi perpend.* GE, *ac biſecetur* VE *in* F, *& erecta perpend.* FH *ad idem Axis Latus, fiat* $FH = \frac{1}{4}GE$; *dico quod Circulus, Centro* H, *radio* HA *deſcriptus, Parabolam interſecabit in punctis tribus vel uno* Z; *ad quæ ductæ rectæ* GZ *Curvæ Parabolicæ perpendiculariter inſiſtunt.*

Ut autem tres ſint hujuſmodi interſectiones, oportet centrum circuli H *ita collocari, ut ſit intra ſpatium Paraboloidibus* (*in Fig.* I.) *incluſum; hoc eſt ut* FH *minus quam* $\sqrt{\frac{8}{27}VF^3}$ *ſive* FH^2 *minus quam cubus ex* $\frac{2}{3}UF$: *atq; adeo* $GE = 4FH$ *minor erit quam* $4\sqrt{\frac{8}{27}VF^3}$ *ſive* $4\sqrt{\frac{1}{27}VE^3}$, *hoc eſt quadratum ex* GE *minor erit quam* $\frac{16}{27}VE^3$. *Coincidunt itaq; hi limites cum Paraboloidibus duabus ejuſdem generis cum iis quibus in Cubicis uſi ſumus, ſed quarum Latus rectum duplo minor eſt;* viz. $\frac{27}{16}$ *Lateris recti Parabolæ, hoc eſt* $\frac{27}{8}$ *ipſius* AV: *ideoq; ea ipſa eſt linea Curva cujus evolutione generatur Parabola, ſic demonſtrante* Hugenio; *quamq; ſemper contingit linea* DF, (*Fig.* II.) *quæ Parabolæ perpendiculariter inſiſtit in puncto* D. *Punctum autem* P, *ſive in quo contingit recta* DF *Paraboloidem, centrum eſt Circuli, qui radio* DP *deſcriptus cum Parabola in puncto* D *coincidit, ſive ejuſdem Curvitatis eſt; ut per ſe ſatis conſtat.*

Deſcriptis itaq; hujuſmodi Paraboloidibus VXP, VNΔ (*Fig.* II.) *utrinq; ab Axe; perſpicuum eſt quod, niſi centrum Circuli conſtituatur intra hos limites, non poſſit ille pluribus quam duobus in punctis Parabolam interſecare: unde determinare licet quibus ſub conditionibus Coefficientes terminorum intermediorum coercentur, in æquationibus Biquadraticis, ut habeantur quatuor radices. Ac prima fronte clarum eſt* p *majorem eſſe non poſſe quam* $\frac{3}{8}bb$ (*ſcil. in formulis ubi habetur* $+p$) *nec* q *quam* $\frac{1}{16}b^3$. *Generaliter vero* $\frac{1}{16}b^3 \mp \frac{1}{4}pb \mp \frac{1}{2}q$, *id eſt diſtantia centri ab Axe* EG, *minor eſſe debet quam* $EH = 4\sqrt{\frac{1}{27}VE^3}$. *hoc eſt* (*ob* $VE = \frac{3}{16}bb \mp \frac{1}{2}p$) *quam* $\frac{1}{4}bb \mp \frac{2}{3}p\sqrt{\frac{1}{16}bb + \text{vel} - p}$; *ſignis* + & — *in dubio relictis, ut ſecundum æquationis cujuſvis naturam variari poſſint; quemadmodum*

admodum in Cubicis ſuperius oſtenſum eſt; ac nollem doctis tædium injicere, aut diſcentibus ſingula particulatim rimandi voluptatem ac exercitationem præripere.

Termini autem ultimi r *limitatio eadem facilitate inveniri nequit; id adeo, quia Problema ſit Solidum, in Curvam Parabolæ demittere perpendicularem, quodq; non ſine ſolutione æquationis Cubicæ reſolvi poſſit. Itaq; prima loco deficiat ſecundus terminus, vel ſi adfuerit, tollatur, ut æquatio habeat formulam* $z^4 . * . pz^2 . qz . r . = 0$. *Ac ſi fuerit* $-r$, *ſemper duabus vel quatuor Radicibus explicari poteſt; ut autem quatuor ſint, oportet centrum circuli intra Paraboloides prædictas conſtitui, ſive ut ſit* $-p$, *ac* qq *minus quam* $\frac{8}{27}p^3$ *ſive cubo ex* $\frac{2}{3}p$. *Deinde habeantur radices æquationis hujus* $y^3 . * . \frac{1}{2}py . \frac{1}{4}q = 0$, *quantitatibus* p & q *iiſdem ſignis annexis quibus in Biquadratica. Hæ autem Radices auxilio Tabulæ Sinuum ſatis expedite inveniuntur. Inventis autem tribus illis* y, (*quæ ſunt ordinatim applicatæ ad Axem Parabolæ, de punctis ubi incidunt perpendicula in Curvam ejus. ſcil.* ZY *in Fig.* III.) $pyy - 3y^4$ *ex minore* y, *quantitatem maximam* r *deſignabit, ſi fuerit* $-r$; *qua ſi minor fuerit* r, *æquatio quatuor habebit radices, aliter duas. Aſt ſi fuerit* $+r$, *oportebit eam minorem eſſe quam* $3y^4 - pyy$ *ex media* y, *nam ſi major ſit, non niſi duas habere poteſt radices, ſaltem ſi minor ſit* r *quam* $3y^4 - pyy$ *ex maxima* y. *Hac vero ſi major ſit, nulla omnino radice vera explicabilis eſt æquatio. Hi vero iidem limites aliter deſignantur ex quantitate* q, *ſcil.* $\frac{1}{2}qy - y^4$ *in primo caſu,* $y^4 - \frac{1}{2}qy$ *in ſecundo, ac* $y^4 + \frac{1}{2}qy$ *in tertio.*

Fieri autem poteſt ut duæ minores quantitates y *non longe diſtent ab invicem, unde evenit quod utraq; ex perpendicularibus major ſit quam recta* GA, *ſcil. cum* qq *majus ſit quam* $\frac{4}{27}p^3$, *minus vero quam* $\frac{8}{27}p^3$; *cadente centro intra ſpatium Paraboloidibus utrinſq; Figuræ* I & II *interjectum. Hoc in caſu, ſi fuerit* $+r$, *non niſi duæ poſſunt eſſe radices, exiſtente* $y^4 + \frac{1}{2}qy$ *ex maxima* y, *major quam* r; *aliter nulla. At ſi* $\frac{1}{2}qy - y^4$ *ex minima* y, *major fuerit quam* r *ſigno* $-$ *notata,* r *vero major quam* $\frac{1}{2}qy - y^4$ *ex media* y, *tunc habentur quatuor radices*

dices; at duæ tantum ſi vel major priore vel minor poſteriore inventa ſit r.

Si vero in æquatione fuerit $+ p$, *vel ſi ſit* $- p$ & qq *majus fuerit quam* $\frac{8}{27}p^3$, *æquatio* $y^3 . * . \frac{1}{2}py . \frac{1}{4}q$ *unica tantum explicatur radice* y; *hoc eſt una tantum perpendicularis de centro Circuli demitti poteſt: unde certo concluditur duas tantum radices haberi poſſe in æquatione data, quarum ſumma, ſi fuerit* $-r$, *cum quantitate* r *augetur; at ſi habeatur* $+ r$, *obtenta quantitate* y, *quantitas illa* r *minor eſſe debet quam* $y^4 + \frac{1}{2}qy$; *nam ſi ea major ſit, æquatio propoſita abſurda* & *impoſſibilis eſt*.

Longum & ſuperfluum eſſet omnes hujus cenſus æquationes percurrere, cum ex jam dictis attendenti ſatis evidens ſit, quæ Negativæ, quæ Affirmativæ ſint; atq; quod Radicum harum Limites ex quantitatibus inventis y *petantur. In exemplum vero, quod cuivis in cæteris imitari licet, proponantur indagandi limites ſive conditiones, ſub quibus in Æquatione Biquadratica* 4 *Radices Affirm. dari poſſint. Hoc autem fit quoties centrum circuli* G *ponitur in ſpatio* UPK, (*Fig*. II.) *ac ſimul habetur* $+ r$ *ſive Circuli radius minor quam* GD: *Unde patet, æquationem de qua agitur hujus eſſe formulæ* $z^4 - bz^3 + pz^2 - qz + r = 0$; p *vero majorem eſſem non poſſe quam* $\frac{3}{8}bb$, *nec* $\frac{1}{4}pb$ *hoc in caſu, quam* $\frac{1}{16}b^3 + \frac{1}{2}q$; *deinde opus eſt ut* $\frac{1}{4}pb - \frac{2}{3}p$ *in* $\sqrt{\frac{1}{16}bb - \frac{1}{6}p}$ *major ſit quam* $\frac{1}{16}b^3 + \frac{1}{2}q - \frac{1}{4}pb$; & *ex his limitibus certo conſtabit centrum intra ſpatium* UPK *inveniri. Ut vero definiatur quantitas* r, *ſolvenda primum eſt Cubica* $y^3 . * . - \frac{3}{16}b^2 - \frac{1}{2}py = \frac{1}{32}b^3 + \frac{1}{4}q - \frac{1}{8}pb$; & *habebuntur puncta, in quæ perpendiculares de centro in Curvam Parabolæ cadunt*.

Inventis autem tribus valoribus hujus y, r *minor eſſe debet quam* $\frac{3}{256}b^4 + \frac{1}{4}bq - \frac{1}{16}bbp + 3y^4 - \frac{3}{8}b^2yy + pyy$ *ex media* y, *major vero quam* $\frac{3}{256}b^4 + \frac{1}{4}bq - \frac{1}{16}bbp + 3y^4 - \frac{3}{8}b^2yy + pyy$ *ex minima* y. *Hos vero limites ſi excedat* r, *non niſi duæ Radices haberi poſſunt. Deniq; ſi* $\frac{3}{256}b^4 + \frac{1}{4}bp - \frac{1}{16}bbp + 3y^4 - \frac{3}{8}bbyy + pyy$ *ex maxima* y, *minor fuerit quam* r, *æquatio propoſita impoſſibilis eſt*.

Accidit etiam ut quatuor ſint Affirmativæ, cum Centrum G *conſtituitur in ſpatiolo* V T S; *ducta ſcil.* R T S *perpendiculari in medium ſuppoſitæ lineæ* A D: *Hoc autem fit cum* p *major eſt quam* $\frac{5}{16}bb$, *ac* $\frac{1}{4}bb-\frac{2}{3}p\sqrt{\frac{1}{16}bb-\frac{1}{6}p}$ *major quam* $\frac{1}{8}pb-\frac{5}{128}bbb-\frac{1}{2}q$. *Quo in caſu ſemper duæ, aliquando tres ex Radicibus fiunt majores quam* $\frac{1}{4}b$.

Notandum vero hic limitem illum ex minima y *productum, aliquando negativum fieri, ſive minorem nihilo; quoties ſcil. maxima ex tribus perpendicularibus major eſt quam* G D. (*Fig.* II.) *Hoc ſi acciderit quantitas* $+r$ *à limite præſcripto ex media* y, *in nihilum minui poteſt. Defectus vero limitis ex minima* y *monſtrat quanta poſſit eſſe* $-r$ *in æquatione, ſi habeantur tres radices Affirmativæ ac una Negativa; quam ſi excedat, non niſi duæ, altera Affirmativa, altera Negativa, dari poſſunt. Hæc autem omnia demonſtrantur ex eo quod prædicti limites quantitatis* r, *ſint differentiæ Quadratorem lineæ* G D *& perpendicularium in Curvam Parabolæ.*

Ob perplexas vero cautiones, quas parit in æquationibus hiſce ſignorum diverſitas, præſtat ſemper ſecundum terminum tollere, ac deinde juxta præcepta jam tradita radicum numerum ac ſigna inquirere; præſertim ſi quantitates illæ y *non multum diſtent ab invicem. Ex quatuor autem hiſce radicibus Affirmativis, duæ ſemper ſunt minores quam* $\frac{1}{4}b$, *duæ vero majores; nempe ſi* D G *minor ſit quam* A G, *ſive* $\frac{1}{4}pb$ *quam* $\frac{3}{64}b^3+q$. *Tres autem minores ſunt quam* $\frac{1}{4}b$, *quoties perpendicularis media, ſive ex media* y *inventa, major eſt quam* A G, *ſive* $\frac{3}{8}bby$ *major quam* $3y^3-pyy$ *ex eadem media* y; *Quarta vero & maxima radix major eſt quam maxima* $y+\frac{1}{4}b$; *æquatuor autem differentiæ ipſius* b *& ſummæ cæterarum trium radicum, ideoq; minor eſt* b. *Sed jam* Manum de Tabula; *Fortaſſis illi qui naturam Parabolæ penitius perſpectam habent, majori compendio hæc omnia peragere valebunt; at ſi quantitates hæ omnes* b. p. q. *&* r, *abſq; reſolutione Cubicæ æquationis rite determinari poſſint, non ſine cauſa ambigitur; quæcunq; enim æquationibus planis hac in re fiunt, non veros limites, ſed approximationes tantum exhibent.*

𝒩. 190.

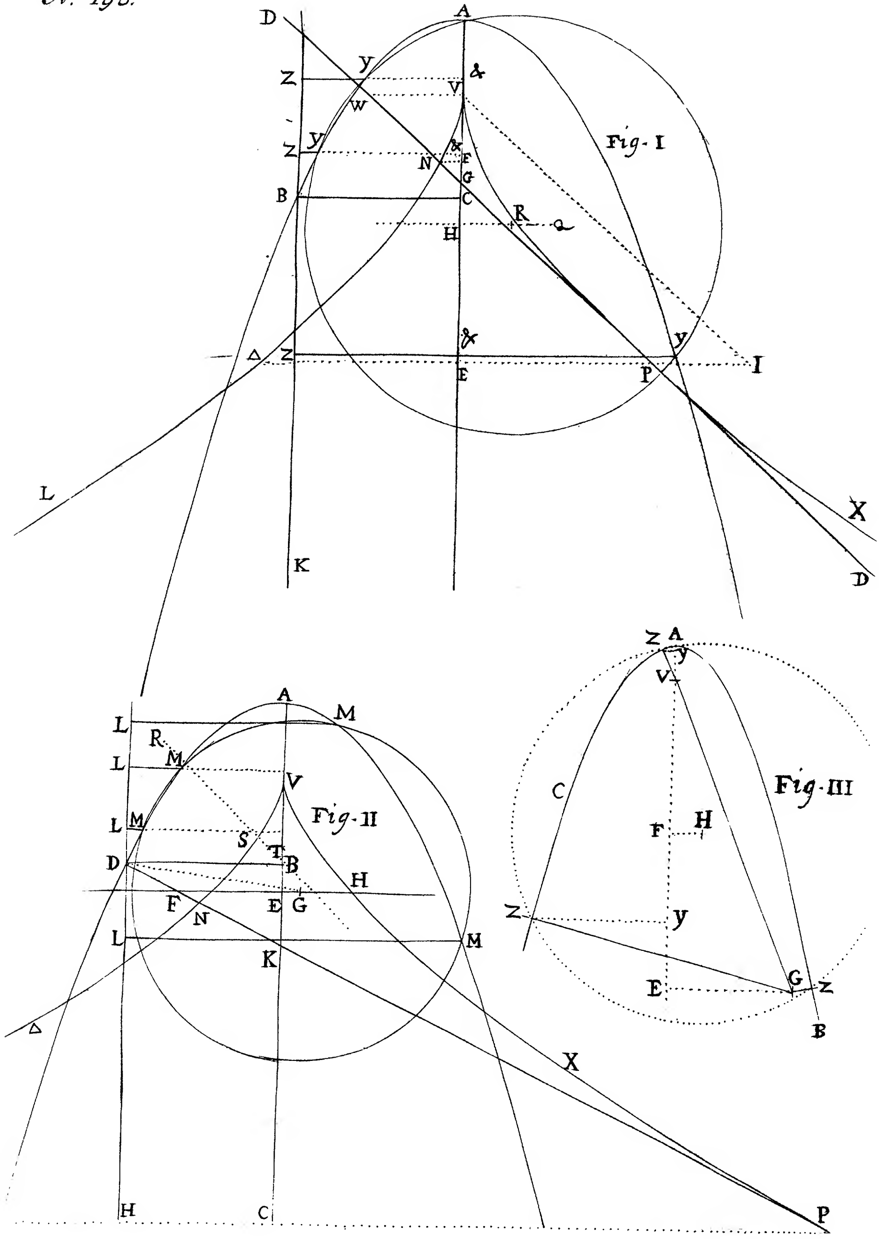
Fig. I
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Fig. II
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Fig. III
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